# 강의 요약서

# A Short Lecture Note

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# Chapter 2. The Fourier Transform

## 2.1 The Fourier Integral

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt$$

$f$ : frequency

$t$ : time

$h(t)$ : function of time. lower case.

$H(f)$ : function of frequency, upper case

$$H(f) = R(f) + jI(f) = |H(f)| e^{j\theta(f)}$$

$R(f)$ : real part

$I(f)$ : imaginary part

$|H(f)|$ : amplitude or Fourier spectrum of $h(t)$

$\theta(f)$ : phase

In general, both $h(t)$ and $H(f)$ are complex quantity.

**Example 2.1 Exponential Waveform**

## 2.2 The Inverse Fourier Transform

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi f t} df$$

Fourier transform pair:

$$h(t) \Leftrightarrow H(f)$$

**Example 2.2 Inverse Fourier Transform of Example 2.1**

## 2.3 Existence of the Fourier Integral

*Condition 1*: If $h(t)$ is integrable in the sense

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

then its Fourier transform $H(f)$ exists and satisfies the inverse Fourier transform.

**Example 2.3** Symmetrical Pulse Waveform

Box Function ⇔ Sinc Function

**Example 2.4** General Pulse Time Waveform

Time shifting ⇔ Linear phase ramp

Linear phase ramp ⇔ Frequency shifting

*Condition 2*: If $h(t) = \beta(t)\sin(2\pi ft + \alpha)$, if $\beta(t+k) < \beta(t)$, and if for $|t| > 0$, the function $h(t)/t$ is absolutely integrable, then $H(f)$ exists and satisfies the inverse Fourier transform.

**Example 2.5** Pulse Frequency Waveform

Sinc Function ⇔ Box Function

*Condition 3*: All functions that hold condition1 and 2 within a certain finite time interval.

**Impulse function:** Any function $\delta(t)$ that satisfies

$$\int_{-\infty}^{\infty} \delta(t - t_0) x(t) dt = x(t_0).$$

**Example 2.6** Impulse function

$$\delta(t - t_0) = \int_{-\infty}^{\infty} e^{j2\pi f(t - t_0)} df$$

$$\delta(t) \Leftrightarrow 1$$

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**Example 2.7** Periodic Functions

$$A\cos(2\pi f_0 t) \Leftrightarrow \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0)$$

**Example 2.8** Sequence of Impulse Functions

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$

**Inversion Formula Proof**

## 2.4 Alternate Fourier Transform Definitions

$$H(\omega) = a_1 \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \qquad \omega = 2\pi f$$

$$h(t) = a_2 \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega$$

where $a_1 a_2 = 1$. For example:

$a_1 = 1, \quad a_2 = 1/2\pi$

$a_1 = a_2 = 1/\sqrt{2\pi}$

$a_1 = 1/2\pi, \quad a_2 = 1$

## 2.5 Fourier Transform Pairs

## PROBLEMS

# Chapter 3. Fourier Transform Properties

## 3.1 Linearity

$$ax(t) + by(t) \Leftrightarrow aX(f) + bY(f)$$

## 3.2 Symmetry

$$H(t) \Leftrightarrow h(-f)$$

## 3.3 Time and Frequency Scaling

$$h(kt) \Leftrightarrow \frac{1}{|k|} H\left(\frac{f}{k}\right)$$

$$\frac{1}{k} h\left(\frac{t}{k}\right) \Leftrightarrow H(kf)$$

## 3.4 Time and Frequency Shifting

$$h(t - t_0) \Leftrightarrow H(f) e^{-j2\pi f t_0}$$

$$h(t) e^{j2\pi t f_0} \Leftrightarrow H(f - f_0)$$

## 3.5 Alternate Inversion Formula

$$h(t) = \left[ \int_{-\infty}^{\infty} H^*(f) e^{-j2\pi f t} df \right]^*$$

## 3.6 Even and Odd Functions

See Table 3.1

## 3.7 Waveform Decomposition

$$\begin{aligned} h(t) &= \left[ \frac{h(t)}{2} + \frac{h(-t)}{2} \right] + \left[ \frac{h(t)}{2} - \frac{h(-t)}{2} \right] \\ &= h_e(t) + h_o(t) \end{aligned}$$

## 3.8 Complex Time Functions

See Table 3.1

Table 3.1 Properties of the Fourier Transform for Complex Functions

| Time Domain $h(t)$ | Frequency Domain $H(f)$ |
|---|---|
| Real | Real part even |
| | Imaginary part odd |
| Imaginary | Real part odd |
| | Imaginary part even |
| Real even, imaginary odd | Real |
| Real odd, imaginary even | Imaginary |
| Real and even | Real and even |
| Real and odd | Imaginary and odd |
| Imaginary and even | Imaginary and even |
| Imaginary and odd | Real and odd |
| Complex and even | Complex and even |
| Complex and odd | Complex and odd |

## 3.9 Summary Table of Fourier Transform

See Table 3.2 in the textbook.

## PROBLEMS

### 3.16 Fourier Transform of Derivatives

If $h(t) \Leftrightarrow H(f)$,

$$\frac{dh(t)}{dt} \Leftrightarrow j2\pi f H(f)$$

$$[-j2\pi t]h(t) \Leftrightarrow \frac{dH(f)}{df}$$

# Chapter 4. Convolution and Correlation

## 4.1 Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

## 4.2 Graphical Evaluation of the Convolution Integral

1. Folding: Take the mirror image of $h(\tau)$ about the originate axis to make $h(-\tau)$.
2. Displacement: Shift $h(-\tau)$ by the amount of $t$ to make $h(t-\tau)$.
3. Multiplication: Multiply the shifted function $h(t-\tau)$ by $x(\tau)$ to make $x(\tau)h(t-\tau)$.
4. Integration: The area under $x(\tau)h(t-\tau)$ is the value of the convolution at time $t$.

## 4.3 Alternate Form of the Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$x(t) * h(t) = h(t) * x(t)$ : Convolution is commutative.

## 4.4 Convolution involving Impulse Functions

$$\delta(t-T) * x(t) = \int_{-\infty}^{\infty} \delta(\tau-T)x(t-\tau)d\tau = x(t-\tau)$$

## 4.5 Time-Convolution Theorem

$$h(t) * x(t) \Leftrightarrow H(f)X(f)$$

## 4.6 Frequency-Convolution Theorem

$$h(t)x(t) \Leftrightarrow H(f) * X(f)$$

## 4.7 Correlation Theorem

$$z(t) = \int_{-\infty}^{\infty} x(\tau)h(t+\tau)d\tau$$

- No folding of one of the integrands in correlation.
- If either $x(t)$ or $h(t)$ is an even function, convolution and correlation are equivalent.
- For complex $x(t)$ or $h(t)$, $z(t) = \int_{-\infty}^{\infty} x^*(\tau)h(t+\tau)d\tau$
- $\int_{-\infty}^{\infty} x(\tau)h(t+\tau)d\tau \Leftrightarrow H(f)X^*(f)$

- If $x(t)$ and $h(t)$ are the same function then $z(t)$ is called *autocorrelation*, while *crosscorrelation* otherwise.

# PROBLEMS

## 4.1 Properties of Convolution

- commutative: $x(t) * h(t) = h(t) * x(t)$
- associative: $h(t) * [g(t) * x(t)] = [h(t) * g(t)] * x(t)$
- distributive over addition: $h(t) * [g(t) + x(t)] = h(t) * g(t) + h(t) * x(t)$
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## 4.10 Convolution of Derivatives

If $h(t) = f(t) * g(t)$ then,

$$\frac{dh(t)}{dt} = \frac{df(t)}{dt} * g(t) = f(t) * \frac{dg(t)}{dt}$$

# Chapter 5. Fourier Series and Sampled Waveforms

*Fourier series is a special case of the Fourier integral.*

*The discrete Fourier transform is the Fourier transform of sampled waveforms.*

## 5.1 Fourier Series

A periodic function $y(t)$ with period $T_0$ expressed as a Fourier series is given by:

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)]$$

where $f_0 = 1/T_0$ : fundamental frequency

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} y(t)\cos(2\pi n f_0 t)dt \qquad n = 0,\ 1,\ 2,\ 3, \cdots$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} y(t)\sin(2\pi n f_0 t)dt \qquad n = 0,\ 1,\ 2,\ 3, \cdots$$

Equivalently, the Fourier series can be expressed in exponential form:

$$y(t) = \sum_{-\infty}^{\infty} \alpha_n e^{j2\pi n f_0 t} \qquad n = 0,\ \pm 1,\ \pm 2,\ \cdots$$

$$\alpha_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} y(t) e^{-j2\pi n f_0 t} dt \qquad n = 0,\ \pm 1,\ \pm 2,\ \cdots$$

## 5.2 Fourier Series as a Special Case of the Fourier Integral

The Fourier transform of the periodic function is an infinite sequence of equidistant impulses with amplitude of $H(n/T_0)$

## 5.3 Waveform Sampling

## 5.4 Sampling Theorems

If the sample interval T is chosen equal to one-half the reciprocal of the highest frequency component, aliasing does not occur.

Nyquist frequency: $f_c = 1/T$

# Chapter 6. The Discrete Fourier Transform

*DFT: a special case of the continuous Fourier transform that is amenable to machine computation.*

Discrete Fourier Transform

$$G\left(\frac{n}{NT}\right)=\sum_{0}^{N-1} g(kT)e^{-j2\pi nk/N} \text{ , n=0, 1, 2, …, N-1}$$

Discrete Inverse Fourier Transform

$$g(kT)=\frac{1}{N}\sum_{n=0}^{N-1} G\left(\frac{n}{NT}\right)e^{j2\pi nk/N} \text{ , k=0, 1, 2, …, N-1}$$

See Figures to understand the relationship between the continuous and discrete Fourier transform.

Discrete Fourier Transform Properties (intuitively similar to continuous Fourier Transform. See Ch. 3.)

- Linearity
- Symmetry
- Time Shifting
- Frequency Shifting
- Even Functions
- Odd Functions
- Waveform Decomposition
- Complex Time Functions
- Time-Convolution Theorem
- Frequency-Convolution Theorem
- Correlation Theorem

# Chapter 7. Discrete Convolution and Correlation

7.1 Discrete Convolution

$$y(kT) = x(kT) * h(kT) = T\sum_{i=0}^{N-1} x(iT)h[(k-i)T]$$

where both $x(kT)$ and $h(kT)$ are periodic functions with period N.

Discrete convolution yields periodic results due to periodicity of the functions being involved. This is called *circular convolution*.

7.4 Discrete Correlation

$$z(kT) = \sum_{i=0}^{N-1} x(iT)h[(k+i)T]$$