1. WHAT IS POLARIZATION?

1.1 Propagation of a monochromatic plane electromagnetic wave

1.1.1 Equation of propagation

The time-space behavior of electromagnetic waves is ruled by the Maxwell equations set defined as

$$\begin{array}{ll} \vec{\nabla} \wedge \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t} & \vec{\nabla} \wedge \vec{H}(\vec{r},t) = \vec{J}_T(\vec{r},t) + \dfrac{\partial \vec{D}(\vec{r},t)}{\partial t} \\ \vec{\nabla} \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t) & \vec{\nabla} \cdot \vec{B}(\vec{r},t) = 0 \end{array} \tag{1}$$

where $\vec{E}(\vec{r},t)$, $\vec{H}(\vec{r},t)$, $\vec{D}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ are the wave electric field, magnetic field, electric induction and magnetic induction respectively.

The total current density, $\vec{J}_T(\vec{r},t) = \vec{J}_a(\vec{r},t) + \vec{J}_c(\vec{r},t)$ is composed of two terms. The first one, $\vec{J}_a(\vec{r},t)$, corresponds to a source term, whereas the conduction current density, $\vec{J}_c(\vec{r},t) = \sigma \vec{E}(\vec{r},t)$, depends on the conductivity of the propagation medium, σ. The scalar field $\rho(\vec{r},t)$ represents the volume density of free charges.

The different fields and induction are related by the following relations

$$\vec{D}(\vec{r},t) = \varepsilon \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t) \quad \vec{B}(\vec{r},t) = \mu\left(\vec{H}(\vec{r},t) + \vec{M}(\vec{r},t)\right) \tag{2}$$

The vectors $\vec{P}(\vec{r},t)$ and $\vec{M}(\vec{r},t)$ are called polarization and magnetization, while ε and μ stand for the medium permittivity and permeability.

In the following, we shall consider the propagation of an electromagnetic wave in a linear medium (free of saturation and hysteresis), free of sources. These hypothesis imposes that $\vec{M}(\vec{r},t) = \vec{P}(\vec{r},t) = \vec{0}$ and $\vec{J}_a(\vec{r},t) = \vec{0}$.

The equation of propagation is found by inserting (1) and (2) into $\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{E}(\vec{r},t)) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}(\vec{r},t)) - \Delta\vec{E}(\vec{r},t)$ and is formulated as

$$\Delta\vec{E}(\vec{r},t) - \mu\varepsilon \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}(\vec{r},t)}{\partial t} = -\frac{1}{\varepsilon} \frac{\partial \vec{\nabla}\rho(\vec{r},t)}{\partial t} \tag{3}$$

1.1.2 Monochromatic plane wave solution

Among the infinite number of solutions to the equation of propagation mentioned in (3), we will study the special case of constant amplitude monochromatic plane waves which is adapted to the analysis of a wave polarization.

The monochromatic assumption implies that the right hand term of (3) is null $\frac{\partial \vec{\nabla}\rho(\vec{r},t)}{\partial t} = \vec{0}$, i.e. the propagation medium is free of mobile electric charges (e.g. is not a plasma whose charged particles may interact with the wave).

The propagation equation expression can be significantly simplified by considering the complex expression, $\underline{\vec{E}}(\vec{r})$, of the monochromatic time-space electric field, $\vec{E}(\vec{r},t)$, defined as

$$\vec{E}(\vec{r},t) = \Re\left(\underline{\vec{E}}(\vec{r})\, e^{j\omega t}\right) \tag{4}$$

The propagation equation mentioned in (3) may then be rewritten as

$$\begin{gathered} \Delta\underline{\vec{E}}(\vec{r}) + \omega^2 \mu\varepsilon\left(1 - j\frac{\sigma}{\varepsilon\omega}\right)\underline{\vec{E}}(\vec{r}) = \Delta\underline{\vec{E}}(\vec{r}) + \underline{k}^2\,\underline{\vec{E}}(\vec{r}) \\ \text{with } \underline{k} = \frac{\omega}{v}\sqrt{1 - j\frac{\sigma}{\varepsilon\omega}} \end{gathered} \tag{5}$$

Here appears the concept of complex dielectric constant

$$\underline{\varepsilon} = \varepsilon' - j\varepsilon'' = \varepsilon - j\frac{\sigma}{\omega} \text{ then } \underline{k} = \frac{\omega}{v}\sqrt{1 - j\frac{\varepsilon''}{\varepsilon}} = \beta - j\alpha \tag{6}$$

In a general way, a monochromatic plane wave , with constant complex amplitude, $\underline{\vec{E}}_0 = E_{\grave{a}} e^{j\vec{\delta}}$, propagating in the direction of the wave vector, $\hat{k}$, has the complex following form

$$\underline{\vec{E}}(\vec{r}) = \underline{\vec{E}}_0 e^{-j\underline{\vec{k}}\cdot\vec{r}} \text{ with } \underline{\vec{E}}(\vec{r})\cdot\hat{k} = 0 \tag{7}$$

One may verify that such a wave satisfies the propagation equation given in (5). Without any loss of generality, the electric field may be represented in an orthonormal basis $(\hat{x},\hat{y},\hat{z})$ defined so that the direction of propagation $\hat{k} = \hat{z}$. The expression of the electric field becomes

$$\underline{\vec{E}}(z) = \underline{\vec{E}}_0 e^{-\alpha z} e^{-j\beta z} \text{ with } \underline{E}_{0z} = 0 \tag{8}$$

It may be observed from (8) that β acts as the wave number in time domain, while α corresponds to an attenuation factor. Back to time domain, this expression becomes in vectorial form

$$\vec{E}(z,t) = \begin{bmatrix} E_{0x} e^{-az}\cos(\omega t - kz + \delta_x) \\ E_{0y} e^{-az}\cos(\omega t - kz + \delta_y) \\ 0 \end{bmatrix} \tag{9}$$

The attenuation term is common to all the elements of the electric field vector and is then unrelated to the wave polarization. For this reason, the medium is assumed to be loss free, $\alpha = 0$, in the following

$$\vec{E}(r,t) = \begin{bmatrix} E_{0x}\cos(\omega t - kz + \delta_x) \\ E_{0y}\cos(\omega t - kz + \delta_y) \\ 0 \end{bmatrix} \tag{10}$$

1.1.3 Spatial evolution of a plane wave vector: helicoidal trajectory

At a fixed time, $t = t_0$, the electric field is composed of two orthogonal sinusoidal waves with, in general, different amplitudes and phases at the origin.

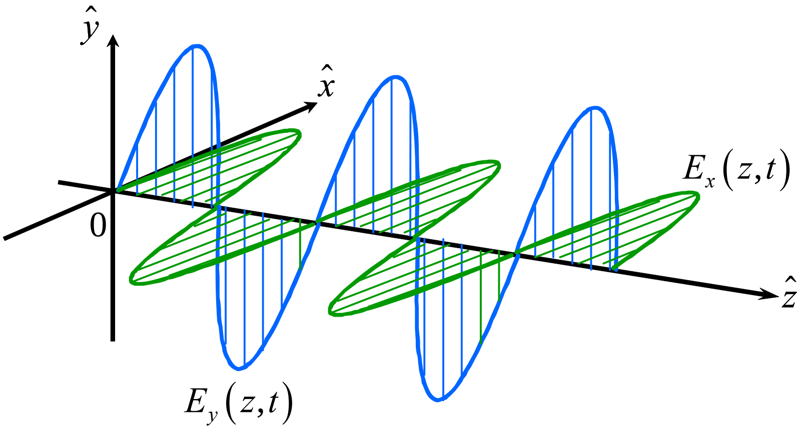


Figure 1 Spatial evolution of a monochromatic plane wave components.

Three particular cases are generally discriminated:

- Linear polarization: $\delta = \delta_y - \delta_x = 0 + m\pi$

The electric field is then a sine wave inscribed within a plane oriented with an angle ϕ with respect to $\hat{x}$

$$\vec{E}(z_0,t) = \sqrt{E_{0x}^2 + E_{0y}^2}\begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}\cos(\omega t_0 - kz + \delta_x) \tag{11}$$

What is Polarization?

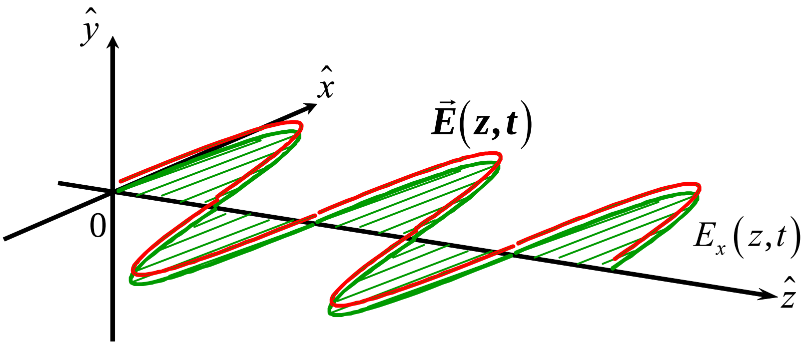


Figure 2 Spatial evolution of a linearly (horizontal) polarized plane wave.

- Circular polarization: $\delta = \delta_y - \delta_x = 0 + m\pi/2$ and $E_{0x} = E_{0y}$

In this case, the wave has a constant modulus and is oriented with an angle $\phi(z)$ with respect to the $\hat{x}$ axis

$$\left|\vec{E}(z,t_0)\right| = E_{0x}^2 + E_{0y}^2 \text{ and } \phi(z) = \pm(\omega t_{\grave{a}} - kz + \delta_x) \quad (12)$$

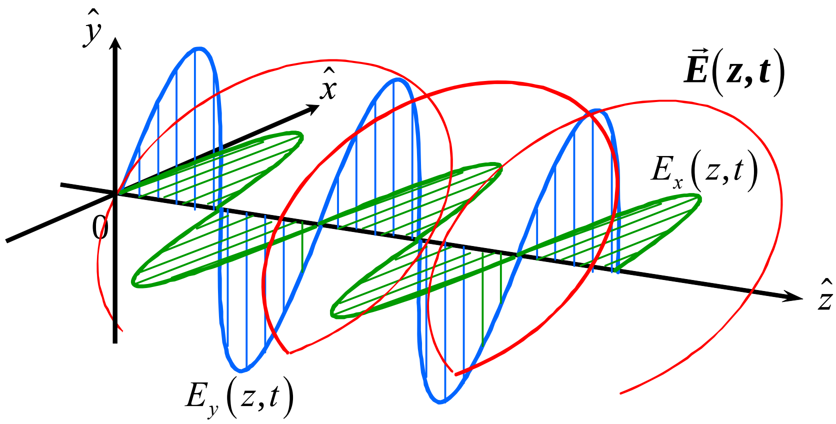


Figure 3 Spatial evolution of a circularly polarized plane wave.

The wave rotates circularly around the $\hat{z}$ axis.

- Elliptic polarization: Otherwise

The wave describes a helicoidal trajectory around the $\hat{z}$ axis.

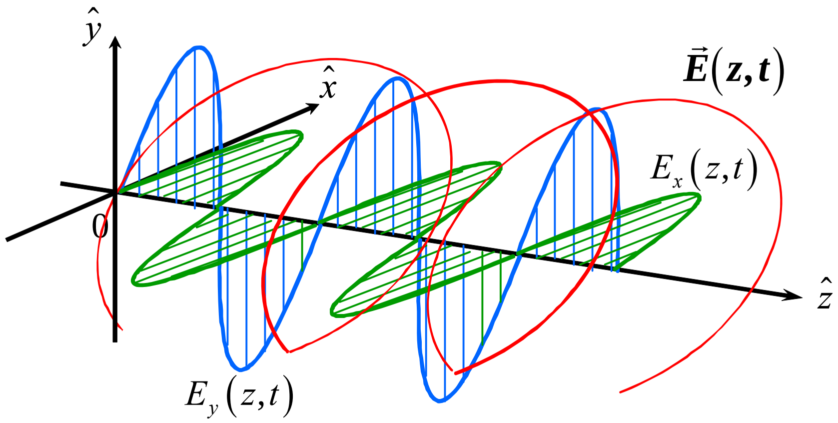


Figure 4 Spatial evolution of a elliptically polarized plane wave.

1.2 Polarization ellipse

1.2.1 Geometrical description

The former paragraph introduced the spatial evolution of a plane monochromatic wave and showed that it follows a helicoidal trajectory along the $\hat{z}$ axis. From a practical point of view, three-dimensional helicoidal curves are difficult to represent and to analyze. This is why a characterization of the wave in the time domain, at a fixed position, $z = z_0$ is generally preferred.

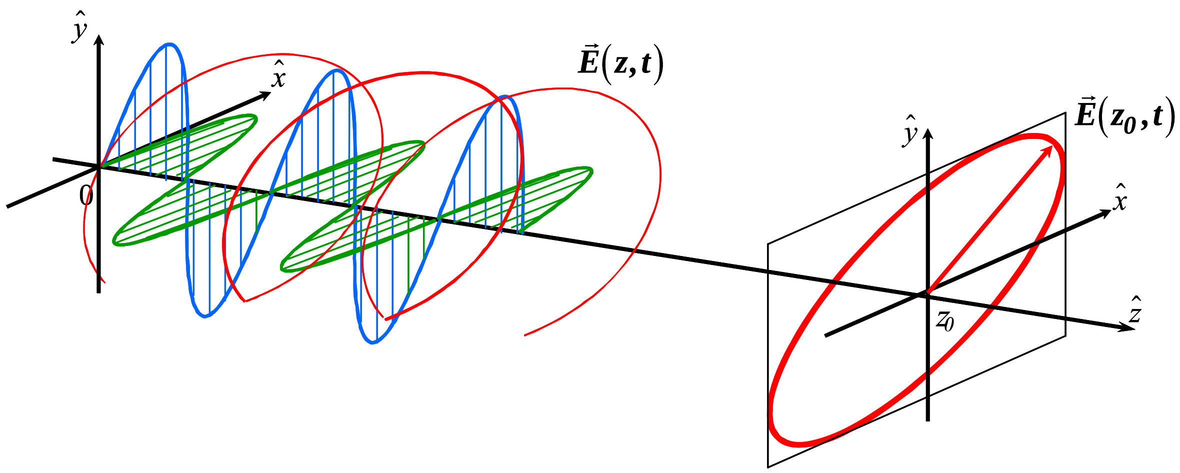


Figure 5 Temporal trajectory of a monochromatic plane wave at a fixed abscissa $z = z_0$.

The temporal behavior is then studied within an equiphase plane, orthogonal to the direction of propagation and at a fixed location along the $\hat{z}$ axis. As time evolves, the wave propagates "through" equi-phase planes nd describe a characteristic elliptical locus as shown in **Figure 5**. The nature of the wave temporal trajectory may be determined from the following parametric relation between the components of $\vec{E}(z_0,t)$

What is Polarization?

$$\left(\frac{E_x(z_0,t)}{E_{0x}}\right)^2 - 2\frac{E_x(z_0,t)E_y(z_0,t)}{E_{0x}E_{0y}}\cos(\delta_y - \delta_x) + \left(\frac{E_y(z_0,t)}{E_{0y}}\right)^2 = \sin(\delta_y - \delta_x) \quad (13)$$

The expression in (13) is the equation of an ellipse, called the polarization ellipse, that describes the wave polarization.

The polarization ellipse shape may be characterized using 3 parameters as shown in **Figure 6**.

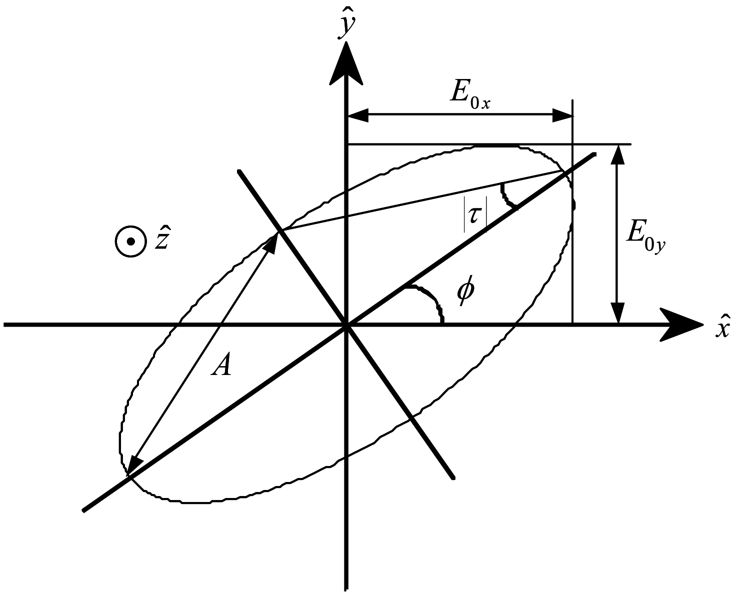


Figure 6 Polarization ellipse.

- A is called the ellipse amplitude and is determined from the ellipse axis as

$$A = \sqrt{E_{0x}^2 + E_{0y}^2} \quad (14)$$

- $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the ellipse orientation and is defined as the angle between the ellipse major axis and $\hat{x}$

$$\tan 2\phi = 2\frac{E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos\delta \text{ with } \delta = \delta_y - \delta_x \quad (15)$$

- $|\tau| \in \left[0, \frac{\pi}{4}\right]$ is the ellipse aperture, also called ellipticity, defined as

$$|\sin 2\tau| = 2\frac{E_{0x}E_{0y}}{E_{0x}^2 + E_{0y}^2}|\sin\delta| \quad (16)$$

1.2.2 Sense of rotation

As time elapses, the wave vector $\vec{E}(z_0,t)$ rotates in the $(\hat{x},\hat{y})$ to describe the polarization ellipse. The time-dependent orientation of $\vec{E}(z_0,t)$ with respect to $\hat{x}$, named $\xi(t)$ is shown in **Figure 7**.

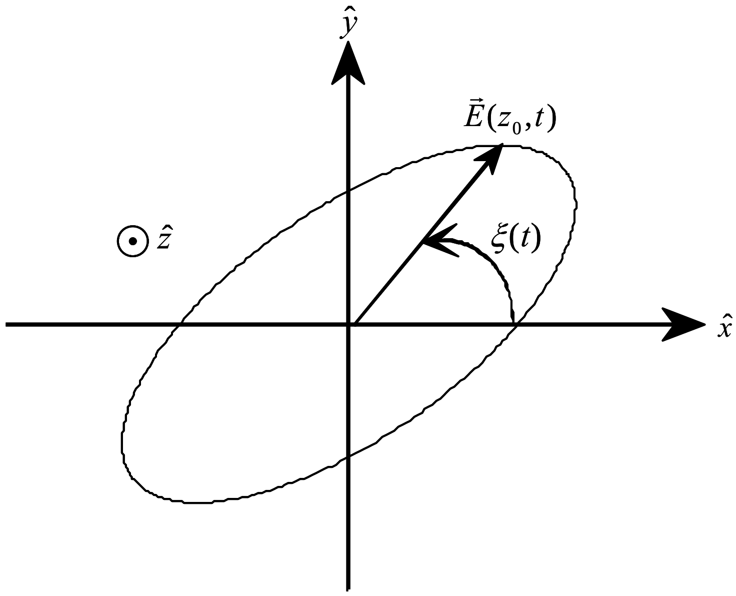


Figure 7 Time-dependent rotation of $\vec{E}(z_0,t)$.

The time-dependent angle may be defined from the components of the wave vector in order to determine its sense of rotation.

$$\tan\xi(t) = \frac{E_y(z_0,t)}{E_x(z_0,t)} = \frac{E_{0y}\cos(\omega t - kz_0 + \delta_y)}{E_{0x}\cos(\omega t - kz_0 + \delta_x)} \tag{17}$$

The sense of rotation may then be related to the sign of the variable τ

$$\frac{\partial\xi(t)}{\partial t} \propto -\sin\delta \Rightarrow sign\left(\frac{\partial\xi(t)}{\partial t}\right) = -sign(\tau) \text{ with } \sin 2\tau = 2\frac{E_{0x}E_{0y}}{E_{0x}^2 + E_{0y}^2}\sin\delta \tag{18}$$

By convention, the sense of rotation is determined while looking in the direction of propagation. A right hand rotation corresponds then to $\frac{\partial\xi(t)}{\partial t} > 0 \Rightarrow (\tau,\delta) < 0$ whereas a left hand rotation is characterized by $\frac{\partial\xi(t)}{\partial t} < 0 \Rightarrow (\tau,\delta) > 0$.

Figure 8 provides a graphical description of the rotation sense convention.

What is Polarization?

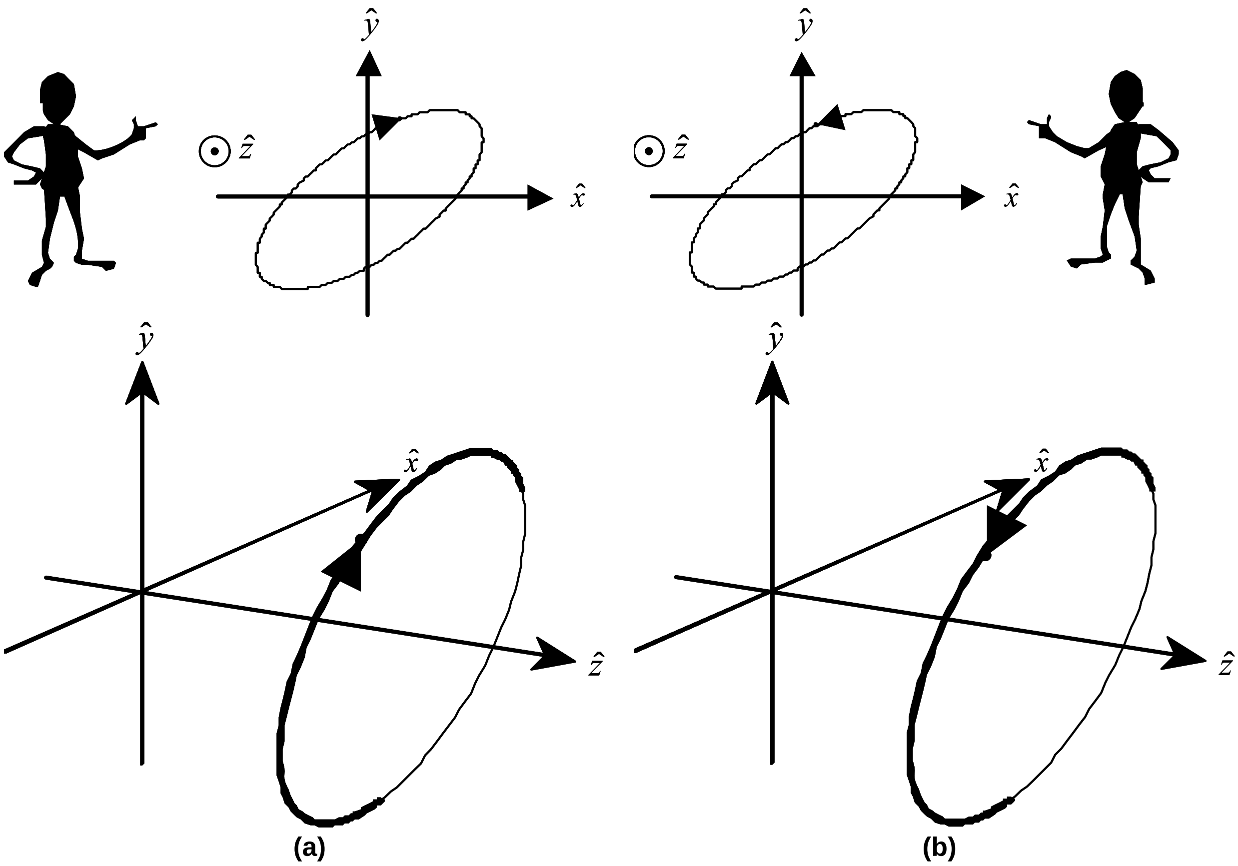


Figure 8 (a) Left hand elliptical polarizations. (b) Right hand elliptical polarizations.

1.2.3 Quick estimation of a wave polarization state

A wave polarization is completely defined by two parameters derived from the polarization ellipse

- its orientation, $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- its ellipticity $\tau \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, with $sign(\tau)$ indicating the sense of rotation

The ellipse amplitude A can be used to estimate the wave power density.

The following procedure provides a quick (calculation free) way to roughly estimate a wave polarization.

Three cases may be discriminated from the knowledge of $\delta = \delta_y - \delta_x$, E_{Ox}, E_{Oy}

- $\delta = 0, \pi$

The polarization is linear since $\tau = 0$ and the orientation angle is given by $\phi = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right)$ if $\delta = 0$ and $\phi = -a\tan\left(\frac{E_{0y}}{E_{0x}}\right)$ if $\delta = \pi$

- $\delta = \pm\frac{\pi}{2}$ and $E_{0x} = E_{0y}$

The polarization is circular, since $\tau = \pm\frac{\pi}{4}$ and the sense of rotation is given by $sign(\delta)$.

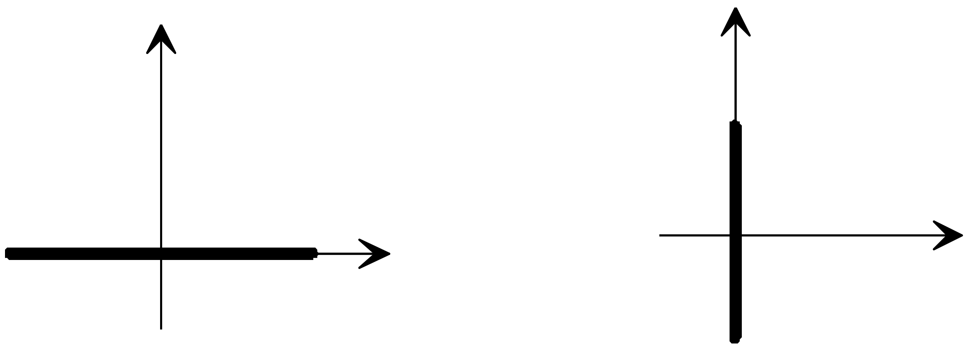
If $\delta < 0$, the polarization is right circular, whereas for $\delta > 0$ the polarization is left circular.

- Otherwise

If $\delta < 0$, the polarization is right elliptic, whereas for $\delta > 0$ the polarization is left elliptic.

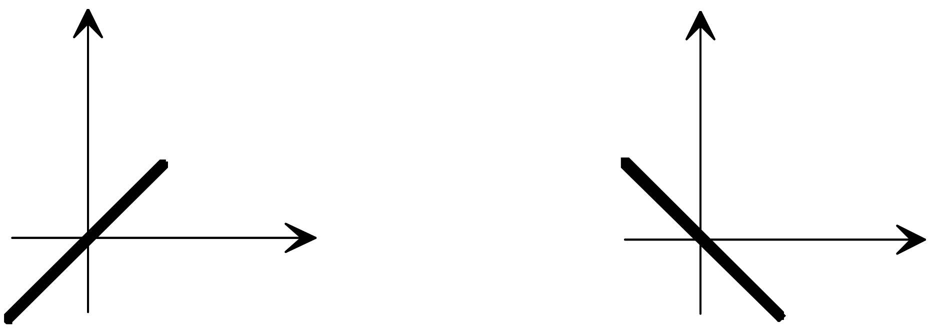
1.2.4 Canonical polarization states

In practice the axes $\hat{x}$ and $\hat{y}$ are generally referred to as the horizontal, $\hat{h}$ and vertical $\hat{v}$ directions.



(a) **(b)**

Figure 9 (a) Horizontal polarization (b) Vertical polarization.



(a) **(b)**

Figure 10 (a) Linear + 45° polarization. (b) Linear - 45° polarization.

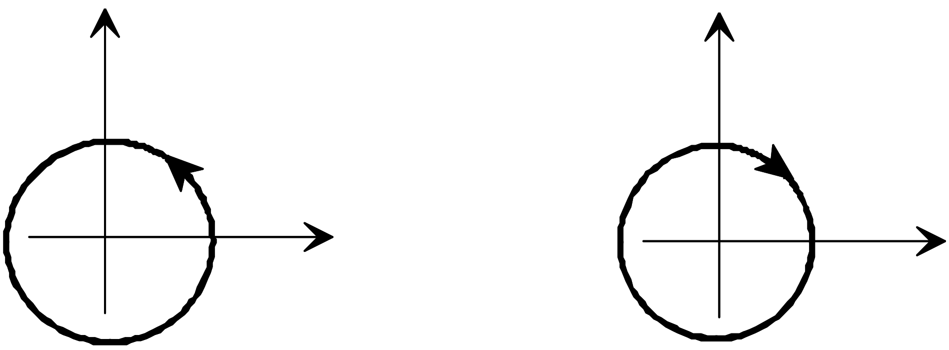


Figure 11 (a) Right circular polarization. (b) Left circular polarization.

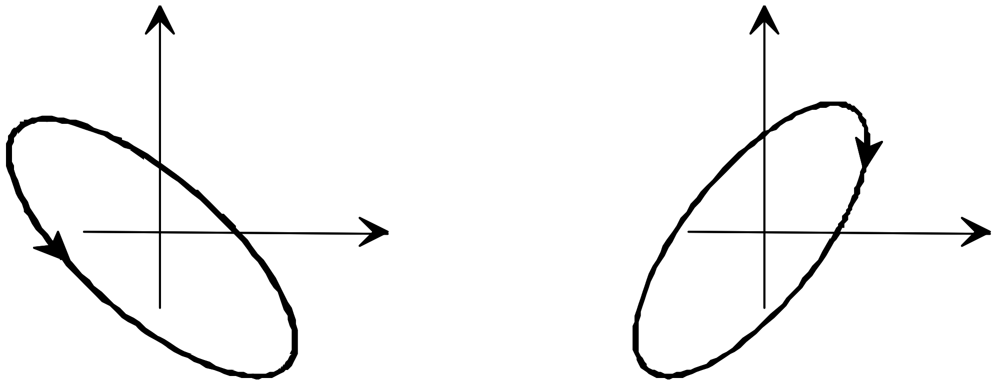


Figure 12 (a) Right elliptical –45 °polarization. (b) Left elliptical +45 °polarization.

1.3 Jones vector

1.3.1 Definition

The representation of a plane monochromatic electric field under the form of a Jones vector aims to describe the wave polarization using the minimum amount of information.

A Jones vector, $\underline{E}$, is defined from the time-space vector $\vec{E}(z,t)$ as

$$\underline{E} = \underline{\vec{E}}(0) \text{ with } \vec{E}(z,t) = \Re\left(\underline{\vec{E}}(z)\, e^{j\omega t}\right) \tag{19}$$

From the formulation of $\vec{E}(z,t)$ given in (10), $\underline{E}$ can be written as

$$\underline{E} = \begin{bmatrix} E_{0x} e^{j\delta_x} \\ E_{0y} e^{j\delta_y} \end{bmatrix} \tag{20}$$

The definitions of a polarization state from the polarization ellipse descriptors or from a Jones vector are equivalent.

A Jones vector can be formulated as a two-dimensional complex vector function of the polarization ellipse characteristics as follows :

What is Polarization?

$$\underline{E} = Ae^{j\alpha} \begin{bmatrix} \cos\phi \cos\tau - j\sin\phi \sin\tau \\ \sin\phi \cos\tau + j\cos\phi \sin\tau \end{bmatrix} \tag{21}$$

Where α is an absolute phase term.

The Jones vector may be written under a more effective matrix form

$$\underline{E} = Ae^{j\alpha} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\tau \\ j\sin\tau \end{bmatrix} \tag{22}$$

1.3.2 Orthogonal polarization states and polarization basis

1.3.2.1 Orthogonal Jones vectors

Two Jones vectors, $\underline{E}_1$ and $\underline{E}_2$ are orthogonal if their hermitian scalar product is equal to 0, i.e.

$$\underline{E}_1^{\dagger} \; \underline{E}_2 = 0 \tag{23}$$

with † the transpose conjugate operator.

From the definition of a Jones vector given in (22), it is straightforward to remark that the orthogonality condition implies that ellipse parameters of $\underline{E}_1$ and $\underline{E}_2$ satisfy

$$\phi_2 = \phi_1 + \frac{\pi}{2} \text{ and } \tau_2 = -\tau_1 \tag{24}$$

One may remark that the orthogonality condition does not depend on the absolute phase term of each Jones vector, α_1 and α_2, i.e. if $\underline{E}_1$ and $\underline{E}_2$ are orthogonal, then $\underline{E}_1$ and $\underline{E}_2 e^{j\psi}$ are orthogonal too, for any value of ψ.

1.3.2.2 Polarization basis

According to the definition of a Jones vector from the time-space electric field given in (19), any Jones vector expressed in the orthonormal basis $(\hat{x}, \hat{y})$ as

$$\underline{E} = \underline{E}_x \hat{x} + \underline{E}_y \hat{y} \tag{25}$$

A Jones vector defined in the basis $(\hat{x}, \hat{y})$, $\underline{E}_{(\hat{x},\hat{y})}$ in the may defined from the unitary vector associated to the horizontal direction, $\hat{x}$

$$\underline{E}_{(\hat{x},\hat{y})} = Ae^{j\alpha} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\tau & j\sin\tau \\ j\sin\tau & \cos\tau \end{bmatrix} \hat{x} \tag{26}$$

This expression may be further developed

$$\underline{E}_{(\hat{x},\hat{y})} = A \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\tau & j\sin\tau \\ j\sin\tau & \cos\tau \end{bmatrix} \begin{bmatrix} e^{-j\alpha} & 0 \\ 0 & e^{j\alpha} \end{bmatrix} \hat{x} \tag{27}$$

The orthogonal Jones may be expressed from (24) as

$$\underline{E}_{\perp(\hat{x},\hat{y})} = A\begin{bmatrix} \cos\left(\phi+\dfrac{\pi}{2}\right) & -\sin\left(\phi+\dfrac{\pi}{2}\right) \\ \sin\left(\phi+\dfrac{\pi}{2}\right) & \cos\left(\phi+\dfrac{\pi}{2}\right) \end{bmatrix}\begin{bmatrix} \cos\tau & -j\sin\tau \\ -j\sin\tau & \cos\tau \end{bmatrix}\begin{bmatrix} e^{-j\alpha} & 0 \\ 0 & e^{j\alpha} \end{bmatrix}\hat{x} \tag{28}$$

or

$$\underline{E}_{\perp(\hat{x},\hat{y})} = A\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\tau & j\sin\tau \\ j\sin\tau & \cos\tau \end{bmatrix}\begin{bmatrix} e^{-j\alpha} & 0 \\ 0 & e^{j\alpha} \end{bmatrix}\hat{y} \tag{29}$$

The matrices associated to the ϕ,τ,α angular variables

$$[U(\phi)] = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \quad [U(\tau)] = \begin{bmatrix} \cos\tau & j\sin\tau \\ j\sin\tau & \cos\tau \end{bmatrix} \quad [U(\alpha)] = \begin{bmatrix} e^{-j\alpha} & 0 \\ 0 & e^{j\alpha} \end{bmatrix}\hat{x} \tag{30}$$

belong to the group of (2×2) Special Unitary complex matrices SU(2) and have the following important properties :

- $\left|[U]\right| = +1$
- $[U]^{-1} = [U]^{\dagger}$
- $[U(x)]^{-1} = [U(-x)]$

Two Jones vectors, $\underline{u}$ and $\underline{v}$ with unitary norms, form a polarization basis if they result from the transformation of the $(\hat{x},\hat{y})$ basis

$$\underline{u} = [U(\phi)][U(\tau)][U(\alpha)]\hat{x} \text{ and } \underline{v} = [U(\phi)][U(\tau)][U(\alpha)]\hat{y} \tag{31}$$

Or equivalently if

$$\underline{u} = [U(\phi)][U(\tau)][U(\alpha)]\hat{x} \text{ and } \underline{v} = [U(\phi+\pi)][U(-\tau)][U(\alpha)]\hat{x} \tag{32}$$

It can be remarked that a polarization basis can uniquely defined by a single vector $\underline{u} = [U(\phi,\tau,\alpha)]\hat{x}$, provided that the second element of the basis $\underline{v}$ verifies $\underline{v} = \underline{u}_{\perp}$.

One has to point out that the definition of a polarization basis provided in (31) and (32) requires that both elements of the basis are constructed using the same absolute phase value α. This condition is not necessary for $\underline{u}$ and $\underline{v}$ to be orthogonal but may involve important problems for the analysis of polarimetric response if it is not respected.

Example :

Let $\underline{R}$ be the Jones vector associated to a right circular polarization

$$\underline{R} = [U(\phi=0)][U(\tau=-\frac{\pi}{4})][U(\alpha=0)]\hat{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -j \end{bmatrix} \tag{33}$$

Then the other element of the orthonormal basis is

$$\underline{R}_{\perp} = [U(\phi=+\frac{\pi}{2})][U(\tau=+\frac{\pi}{4})][U(\alpha=0)]\hat{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} -j \\ 1 \end{bmatrix} \tag{34}$$

It can observed that $\underline{R}_\perp$ is slightly different from the usual definition of a left circular polarization Jones vector $\underline{L}$

$$\underline{L} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ j\end{bmatrix} = \underline{R}_\perp e^{j\frac{\pi}{2}} \tag{35}$$

Both Jones vector depict a left circular polarization state but $\underline{R}_\perp$ only may be coupled to $\underline{R}$ to form a polarization in the sense it was defined in (31) and (32).

1.3.3 Polarization ratio

1.3.3.1 Definition

An efficient way to characterize a Jones vector polarization state is to build its polarization ratio defined as

$$\rho = \frac{\underline{E}_y}{\underline{E}_x} = \frac{E_{0y}}{E_{0x}} e^{j(\delta_y - \delta_y)} \tag{36}$$

The polarization ratio may be written as a function of the polarization ellipse parameters as

$$\rho = \frac{\sin\phi\cos\tau + j\cos\phi\ \sin\tau}{\cos\phi\cos\tau - j\sin\phi\ \sin\tau} \tag{37}$$

Canonical polarization states can be easily discriminated from the knowledge of ρ:

- $Arg(\rho) = 0 + m\pi$

The polarization is linear and the orientation angle is given by $\phi = \tan^{-1}(\rho)$

- $\rho = e^{j\frac{\pi}{4}}$

The polarization is circular, $sign(Arg(\rho))$.

If $Arg(\rho) < 0$, the polarization is right circular, whereas for $Arg(\rho) > 0$ the polarization is left circular.

- Otherwise

If $Arg(\rho) < 0$, the polarization is right elliptic, whereas for $Arg(\rho) > 0$ the polarization is left elliptic.

1.3.3.2 Representation of polarization states

Canonical polarization states are given in the following table

Polarization States	Unitary Jones vector $\hat{u}_{(\hat{x},\hat{y})}$	Orientation (ϕ)	Ellipticity (τ)	Polarizatio ratio $\rho_{(\hat{x},\hat{y})}$
Horizontal (H)	$\begin{bmatrix}1\\0\end{bmatrix}$	0	0	0
Vertical (V)	$\begin{bmatrix}0\\1\end{bmatrix}$	$\frac{\pi}{2}$	0	$\propto$
Linear +45°	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{\pi}{4}$	0	1
Linear -45°	$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$	$\frac{3\pi}{4}$	0	-1
Left circular	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$	?	$\frac{\pi}{4}$	j
Right circular	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$	?	$-\frac{\pi}{4}$	-j

A polarization map may also be built from the representation of polarization states in a complex plane.

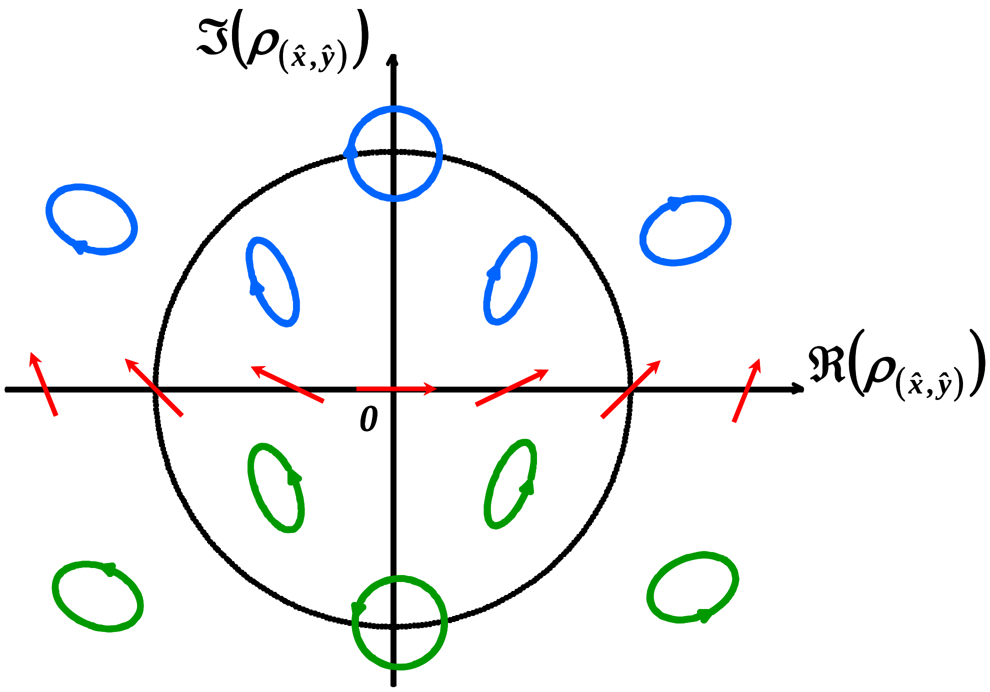


Figure 13 Polarization map in the real and imaginary polarization ratio plane.

1.3.3.3 Orthogonal polarization states and polarization basis

A Jones vector components may be expressed as a function of its polarization as follows

$$\underline{E} = Ae^{j\alpha}\begin{bmatrix}\cos\phi\cos\tau - j\sin\phi\,\sin\tau \\ \sin\phi\cos\tau + j\cos\phi\,\sin\tau\end{bmatrix} = Ae^{j\alpha'}\frac{1}{\sqrt{1+|\rho|^2}}\begin{bmatrix}1 \\ \rho\end{bmatrix} \tag{38}$$

where the absolute phase term is modified in order to account for the polarization ratio argument

$$\alpha' = \alpha + \arg(\cos\phi\cos\tau - j\sin\phi\sin\tau) \tag{39}$$

The orthogonal Jones vector is given by

$$\underline{E}_\perp = Ae^{j\alpha}\begin{bmatrix}-\sin\phi\cos\tau + j\cos\phi\,\sin\tau \\ \cos\phi\cos\tau + j\sin\phi\,\sin\tau\end{bmatrix} = Ae^{j\alpha''}\frac{1}{\sqrt{1+|\rho_\perp|^2}}\begin{bmatrix}1 \\ \rho_\perp\end{bmatrix} \tag{40}$$

with $\alpha'' = \alpha + \arg(-\sin\phi\cos\tau + j\cos\phi\sin\tau)$

With $\rho_\perp$ the orthogonal polarization ratio defined as

$$\rho_\perp = -\frac{1}{\rho^*} \tag{41}$$

A polarization basis $(\hat{u},\hat{v})$ may then be defined from a vector polarization ratio as follows

$$\underline{u} = [U(\rho)][U(\alpha)]\hat{x} \;\text{ and }\; \underline{v} = [U(\rho)][U(\alpha)]\hat{y} \tag{42}$$

with

$$[U(\rho)] = [U(\phi)][U(\tau)] = \frac{1}{\sqrt{1+|\rho|^2}} \begin{bmatrix} 1 & -\rho^* \\ \rho & 1 \end{bmatrix} \tag{43}$$

A polarization basis can uniquely defined by the polarization ration of a single vector $\underline{u}$, provided that the second element of the basis $\underline{v}$ verifies $\underline{v} = \underline{u}_\perp$.

Note that the use of the following transformation matrix

$$\frac{|\rho|}{\sqrt{1+|\rho|^2}} \begin{bmatrix} 1 & 1 \\ \rho & \rho_\perp \end{bmatrix} \tag{44}$$

would lead to the same polarization state for $\underline{v}$, but to different α phase terms for $\underline{u}$ and $\underline{v}$.

1.3.4 Change of polarimetric basis

One of the main advantages of radar polarimetry resides in the fact that once a target response is acquired in a polarization basis, the response in any basis can be obtained from a simple mathematical transformation and does not require any additional measurements.

A Jones vector, $\underline{E}_{(\hat{x},\hat{y})} = \underline{E}_x \hat{x} + \underline{E}_y \hat{y}$ expressed in the $(\hat{x},\hat{y})$ orthonormal polarimetric basis, transforms to $\underline{E}_{(\hat{u},\hat{v})} = \underline{E}_u \hat{u} + \underline{E}_v \hat{v}$ in the $(\hat{u},\hat{v})$ orthonormal basis, with $\hat{u}$ given by $\hat{u} = [U(\phi)][U(\tau)][U(\alpha)]\hat{x}$, by the way of a Special Unitary transformation.

The coordinates $\underline{E}_u$ and $\underline{E}_v$ can be determined according to the following expression

$$\begin{gathered} \underline{E}_{(\hat{u},\hat{v})} = \underline{E}_u \hat{u} + \underline{E}_v \hat{v} \Rightarrow \underline{E}_{(\hat{x},\hat{y})} = \underline{E}_u [U(\phi,\tau,\alpha)]\hat{x} + \underline{E}_u [U(\phi,\tau,\alpha)]\hat{y} = \underline{E}_x \hat{x} + \underline{E}_y \hat{y} \\ \Rightarrow \underline{E}_u = [U(\phi,\tau,\alpha)]^{-1} \underline{E}_x \text{ and } \underline{E}_v = [U(\phi,\tau,\alpha)]^{-1} \underline{E}_y \end{gathered} \tag{45}$$

Finally

$$\begin{gathered} \underline{E}_{(\hat{u},\hat{v})} = [U_{(\hat{x},\hat{y})\to(\hat{u},\hat{v})}]\underline{E}_{(\hat{x},\hat{y})} \text{ with} \\ [U_{(\hat{x},\hat{y})\to(\hat{u},\hat{v})}] = [U(\phi,\tau,\alpha)]^{-1} = [U(-\alpha)][U(-\tau)][U(-\phi)] \end{gathered} \tag{46}$$

Similarly a change of polarimetric basis from $(\hat{a},\hat{b})$ to $(\hat{u},\hat{v})$ can be operated using a transformation matrix as follows

$$\begin{aligned} \hat{a} &= [U(\phi_a,\tau_a,\alpha_a)]\hat{x} = [U_a]\hat{x} \\ \hat{u} &= [U(\phi_u,\tau_u,\alpha_u)]\hat{x} = [U_u]\hat{x} \end{aligned} \Rightarrow \hat{u} = [U_u][U_a]^{-1}\hat{a} \Rightarrow \underline{E}_{(\hat{u},\hat{v})} = [U_a][U_u]^{-1}\underline{E}_{(\hat{u},\hat{v})} \tag{47}$$

Note that transformation matrices can also be built from the polarization ratio as shown in the former paragraph.

1.4 Stokes vector

1.4.1 Real representation of a plane wave vector

In the previous section, we presented the representation of the polarization state of a plane monochromatic electric field by means of the complex Jones vector. As it can be observed in (20), the Jones vector is determined by two complex quantities. Consequently, if the goal of a given system is to measure the Jones vector of the received wave, this system must record the amplitude and the phase of the incoming wave.

The availability of coherent systems able to measure the amplitude and phase of the incoming waves is relatively recent. In the past, only non-coherent systems were available. These systems are only able to measure the power of an incoming wave. Consequently, it was necessary to characterize the polarization of a wave only by power measurements. This characterization is carried out by the so-called Stokes vector.

Given the Jones vector $\underline{E}$ of a given wave, we can create the hermitian product as follows

$$\underline{E} \cdot \underline{E}^{T*} = \begin{bmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{bmatrix} \tag{48}$$

giving as a result a 2×2 hermitian matrix. At this point, if we consider the Pauli group of matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{49}$$

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{50}$$

$$\sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{51}$$

$$\sigma_3 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \tag{52}$$

It is possible to decompose (48) as follows

$$\underline{E} \cdot \underline{E}^{T*} = \frac{1}{2}\left\{ g_0\sigma_0 + g_1\sigma_1 + g_2\sigma_2 + g_3\sigma_3 \right\} = \frac{1}{2}\begin{bmatrix} g_0 + g_1 & g_2 - jg_3 \\ g_2 + jg_3 & g_0 - g_1 \end{bmatrix} \tag{53}$$

where the parameters $\{g_0, g_1, g_2, g_3\}$ receive the name of Stokes parameters. From (53), the Stokes vector, denoted by $\underline{g}_{\underline{E}}$

$$\underline{g}_{\underline{E}} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} |E_x|^2 + |E_y|^2 \\ |E_x|^2 - |E_y|^2 \\ 2\Re\left\{E_x E_y^*\right\} \\ -2\Im\left\{E_x E_y^*\right\} \end{bmatrix} \tag{54}$$

where the following relation can be established

$$g_0^2 = g_1^2 + g_2^2 + g_3^2 \tag{55}$$

The relation given at (55) establishes that in the set {g_0, g_1, g_2, g_3} there are only three independent parameters. The Stokes parameter g_0 is always equal to the total power (density) of the wave; g_1 is equal to the total power in the linear horizontal or vertical polarized components; g_2 is equal to the power in the linearly polarized components at tilt angles ψ=45 degrees or 135 degrees and g_3 is equal to the power in the left-handed and right-handed circular polarized component in the plane wave. If any of the parameters {g_0, g_1, g_2, g_3} has a non-zero value, it indicates the presence of a polarized component in the plane wave.

The Stokes parameters are sufficient to characterize the magnitude and the relative phase, and hence, the polarization of a wave. As it can be observed in (54), the Stokes parameters can be obtained from only power measurements. Consequently, the Stokes vector is capable to characterize the polarization state of a wave by 4 real parameters. The next section presents the relations existing between the Stokes parameters {g_0, g_1, g_2, g_3} and the polarization ellipse parameters.

1.4.2 Relation between the Stokes vector and the polarization ellipse

The Stokes vector given at (54) can be written as follows

$$\underline{g}_E = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos(\delta) \\ 2E_{0x}E_{0y}\sin(\delta) \end{bmatrix} \tag{56}$$

If now, we consider the expression presented at (15) and (16), the Stokes vector can be written as a function of: the polarization ellipse orientation angle ϕ and ellipse aperture angle τ and the polarization ellipse aperture A

$$\underline{g}_E = \begin{bmatrix} A \\ A\cos(2\phi)\cos(2\tau) \\ A\sin(2\phi)\cos(2\tau) \\ A\sin(2\tau) \end{bmatrix} \tag{57}$$

At Section 1.3.2.2, we represented a given Stokes vector as the product of three unitary matrices belonging to the special unitary SU(2), see (27) and (30). By using the existing homorphism between the group SU(2) and the group O(3) of real orthogonal matrices, given by

$$\left[O_3(2\theta)\right]_{p,q} = \frac{1}{2}Tr\left(\left[U_2(\theta)\right]^{T*}\sigma_p\left[U_2(\theta)\right]\sigma_q\right) \tag{58}$$

we can write the Stokes vector of a particular polarization state as follows

$$\underline{g}_{\underline{E}} = A^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\phi) & -\sin(2\phi) & 0 \\ 0 & \sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\tau) & 0 & -\sin(2\tau) \\ 0 & 0 & 1 & 0 \\ 0 & \sin(2\tau) & 0 & \cos(2\tau) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(2\alpha) & -\sin(2\alpha) \\ 0 & 0 & \sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \underline{g}_{\hat{u}_x} \tag{59}$$

where $\underline{g}_{\hat{u}_x}$ represents the Stokes vector associated with the horizontal polarization (59) can be rewritten compactly as

$$\underline{g}_{\underline{E}} = A^2 \left[O_4(2\phi)\right]\left[O_4(2\tau)\right]\left[O_4(2\alpha)\right]\underline{g}_{\hat{u}_x} \tag{60}$$

1.4.2.1 Orthogonal Stokes vectors

At Section 1.3.2.1 we defined the orthogonal Jones vectors. As observed in (24), the orthogonality can be established in terms if the angles defining the polarization ellipse. Consequently, given the Stokes vector of a given polarization state $\underline{g}_{\underline{E}}$, see (57), the orthogonal Stokes vector is

$$\underline{g}_{\underline{E}\perp} = \begin{bmatrix} A \\ -A\cos(2\phi)\cos(2\tau) \\ -A\sin(2\phi)\cos(2\tau) \\ -A\sin(2\tau) \end{bmatrix} \tag{61}$$

1.4.2.2 Canonical polarization states

The Stokes vector for the canonical polarization states are presented in the following formula

Polarization States	Unitary Jones vector $\hat{u}_{(\hat{x},\hat{y})}$	Stokes vector $\underline{g}_{\underline{E}}$
Horizontal (H)	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
Vertical (V)	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

Linear +45°	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$
Linear -45°	$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$
Left circular	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$
Right circular	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$

1.4.3 Representation of Stokes vectors: The Poincaré sphere

As it has been mentioned, the Stokes vector is completely determined by three independent parameters. Consequently, a three dimensional representation of the Stokes vector is possible. This representation receives the name of Poincaré sphere.

If we consider the expression for the Stokes vector at (57), it can be observed that the three parameters $\{g_1, g_2, g_3\}$ can be considered as the spherical coordinates of a point in a sphere of radius g_0. **Figure 14** presents an scheme of this representation. From this figure, it can be clearly observe which is the effect of the polarization ellipse angles ϕ and τ, where the longitude and latitude of the point defining the polarization state are related with 2ϕ and 2τ.

An interesting aspect to highlight about the Poincaré sphere is the representation of orthogonal polarization states. Taking into account the expressions presented at (57) and (61) it can be observed that two orthogonal polarization states are represented by antipodal points in the Poincaré sphere.

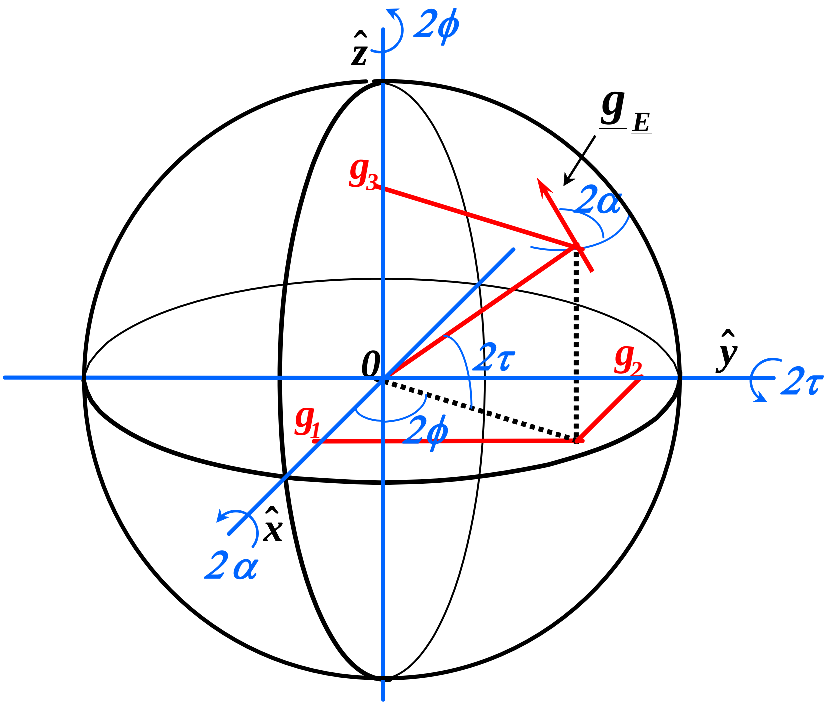


Figure 14 Poincaré sphere.

Finally, **Figure 15** gives the representation of some canonical polarization states within the Poincaré sphere.

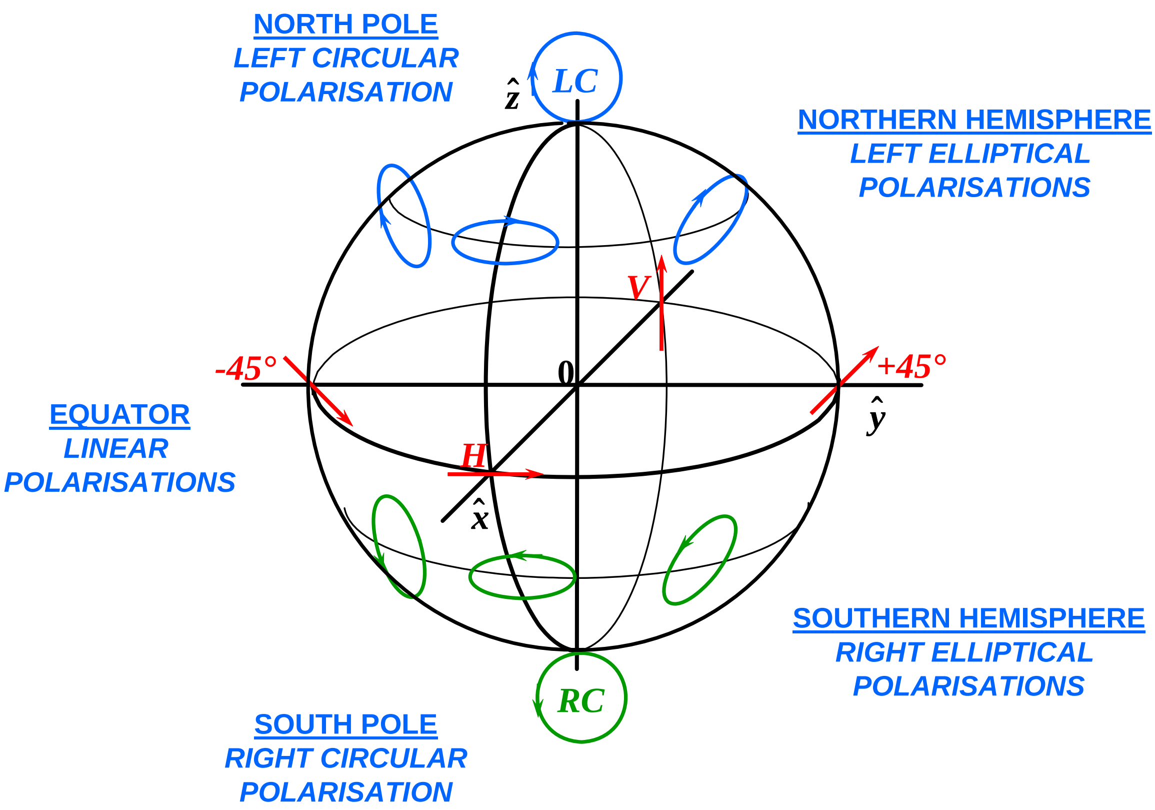


Figure 15 Canonical polarization states represented at the Poincaré sphere.