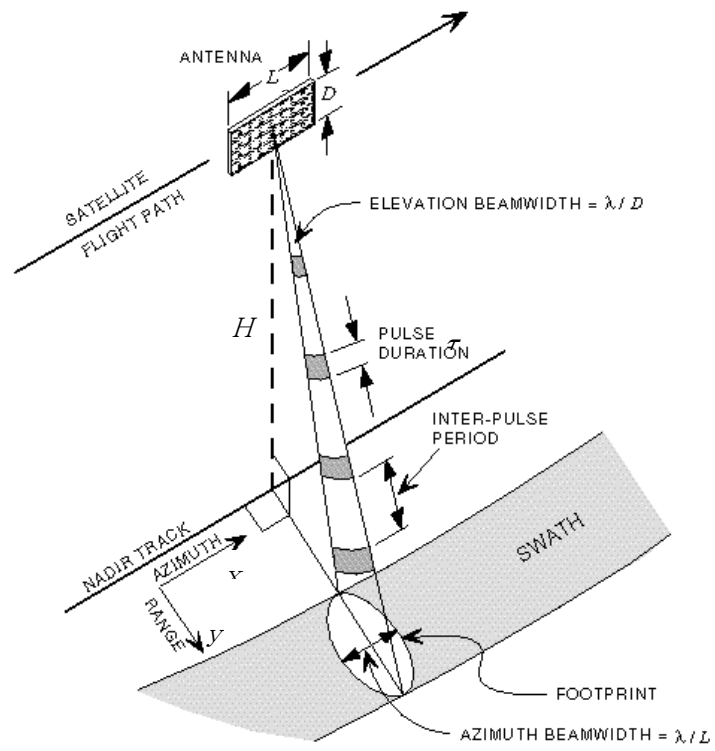


Synthetic Aperture Radar

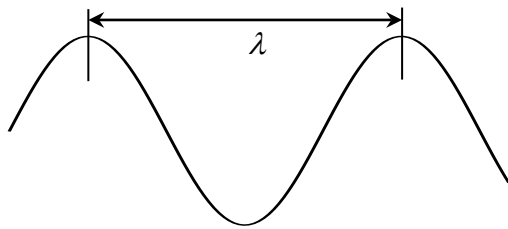
Focusing by Rectangular Algorithm



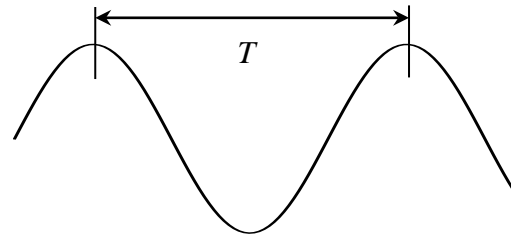
References:

1. Curlander and McDonough, 1991. Synthetic Aperture Radar Systems and Signal Processing, John Wiley & Sons, Chapter 4.
2. ASF Scientific User's Manual

1. Wave



Spatial measurement of wave at a time (photo)



Time measurement of wave at a point

Description of wave:

$$s(\vec{r}, t) = Ae^{j\phi} = A \exp[j(\vec{k} \cdot \vec{r} - \omega t)]$$

\vec{r} : location vector

t : time

A : amplitude of the wave

$$j = \sqrt{-1}$$

ϕ : phase of the wave

$$\text{Frequency: } f \equiv \frac{\dot{\phi}}{2\pi}$$

$$\text{Period: } T \equiv \frac{1}{f}$$

Angular frequency: $\omega = 2\pi f$

\vec{k} : wave number vector

$\vec{k} = (k_x, k_y, k_z)$ in Cartesian coordinate.

Wave number: $k = 2\pi / \lambda$

$$k_x = 2\pi / \lambda_x, \quad k_y = 2\pi / \lambda_y, \quad k_z = 2\pi / \lambda_z$$

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

2. Range Compression

As the vehicle moves along its path, the radar transmits a linear-FM, or “chirp”, pulse

$$s(t) = \exp[j2\pi(f_c t + Kt^2 / 2)], \quad |t| < \tau_p / 2 \quad (1)$$

where, frequency (time derivative of phase) $f = f_c + Kt$, which is a linear function of time over the pulse duration τ_p with chirp rate K .

The chirp bandwidth (in range direction) is $B_R = K\tau_p$ since the frequency starts at $f_c - K\tau_p / 2$, sweeps through all intermediate frequencies, and ends at $f_c + K\tau_p / 2$.

The multi-pulse real transmitted signal is then

$$p(t) = \sum_n s(t - nT_p), \quad (5)$$

where T_p is the pulse repetition period and the sum includes all pulses for which the target is in the radar beam.

At an arbitrary time t , the radar is at some slant range $R(t)$ from a target point with image coordinates (x_c, R_c) at the radar beam centre. The received pulse train is then

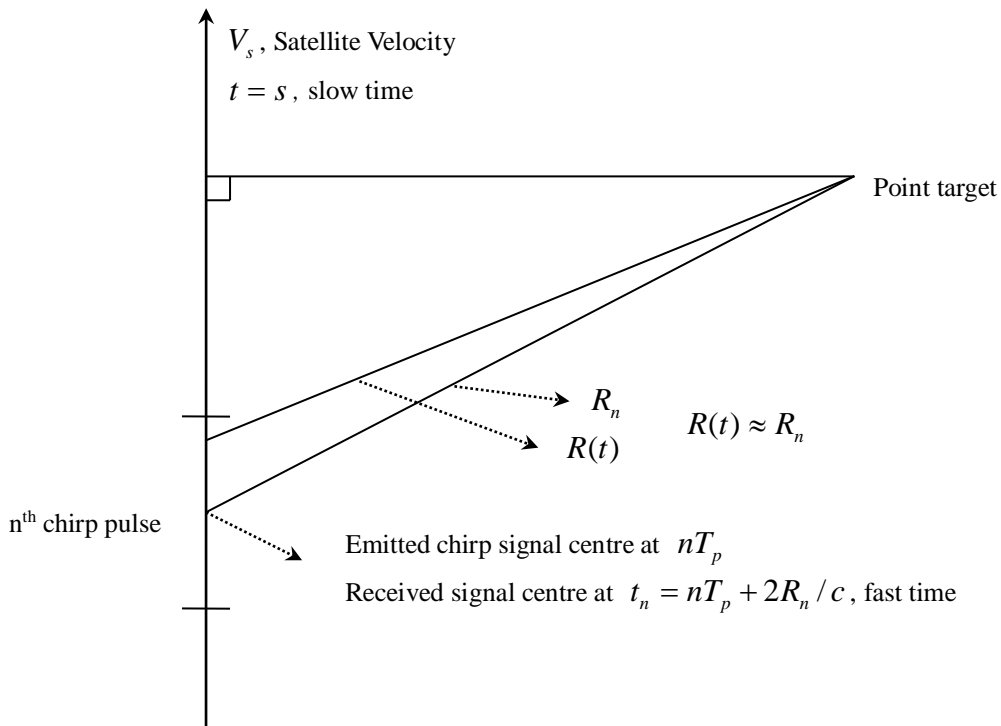
$$v(t) = \sum_n a_n s[t - nT_p - 2R(t)/c], \quad |t - nT_p - 2R(t)/c| \leq \frac{\tau_p}{2}. \quad (6)$$

Range compression is to correlate the received pulse with $s^*(t)$ (equivalently matched filtering with $s^*(-t)$) as:

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} s^*(t' - t)v(t')dt' \\ &= \sum_n a_n \int_{-\infty}^{\infty} s^*(t' - t)s(t' - nT_p - 2R(t')/c)dt' \end{aligned} \quad (7)$$

As shown in the figure below, the variation of $R(t)$ within a chirp pulse is very small, so that we can assume that $R(t) = R_n$, the distance between the sensor and the target at

chirp centre time. This means that the slant range $R(t)$ can be assumed constant during the time of one pulse width. This assumption enables the general two-dimensional compression problem to be de-coupled into a sequence of two one-dimensional compression operations, one in fast time (range) and the other in slow time (azimuth). Since slow time measures a coordinates orthogonal to fast time, this process sequence is



called the **rectangular algorithm**. The centre of the returned chirp signal from a point target is at t_n , where

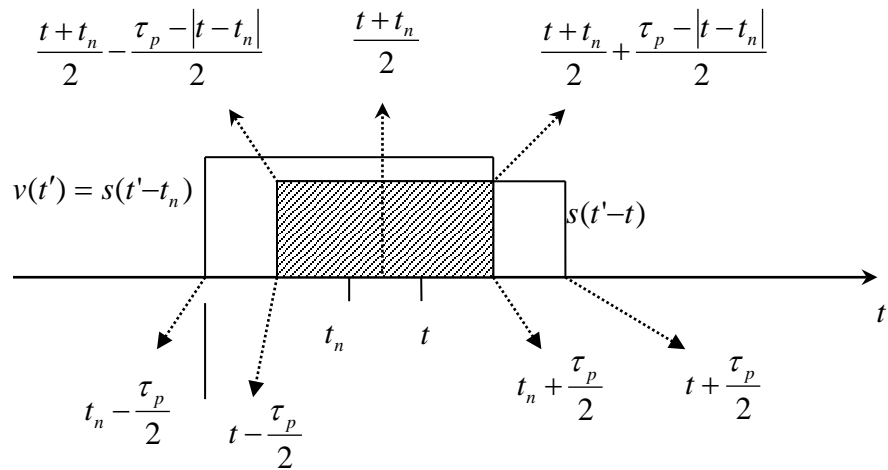
$$t_n = nT_p + 2R_n / c . \quad (8)$$

The matched filtering becomes:

$$g(t) = \sum_n a_n \int_{-\infty}^{\infty} s^*(t'-t)s(t'-t_n)dt' \quad (9)$$

The actual integration is not $(-\infty, \infty)$ as there is a time limit in both $s(t)$ and $v(t)$.

The range of effective integration also depends on t . The range of integration can be obtained from the diagram below.



Considering carefully the limit of the integration gives:

- 1) If $|t - t_n| < \tau_p$, then

$$g(t) = \sum_n a_n \int_{\frac{t+t_n-\tau_p-|t-t_n|}{2}}^{\frac{t+t_n+\tau_p-|t-t_n|}{2}} \exp\left\{-j2\pi\left[f_c(t'-t) + K(t'-t)^2/2\right]\right\} \\ \cdot \exp\left\{j2\pi\left[f_c(t'-t_n) + K(t'-t_n)^2/2\right]\right\} dt'$$

$$2) \quad \text{else } g(t) = 0. \quad (10)$$

We only evaluate the first case.

$$g(t) = \sum_n a_n \exp\left\{-j2\pi\left[-f_c t + Kt^2/2\right]\right\} \cdot \exp\left\{j2\pi\left[-f_c t_n + Kt_n^2/2\right]\right\} \\ \int_{\frac{t+t_n-\tau_p-|t-t_n|}{2}}^{\frac{t+t_n+\tau_p-|t-t_n|}{2}} \exp\left\{-j2\pi\left[f_c t' + K(t'^2 - 2tt')/2\right]\right\} \exp\left\{j2\pi\left[f_c t' + K(t'^2 - 2t_n t')/2\right]\right\} dt' \\ = \sum_n a_n \exp\left\{j2\pi\left[f_c(t-t_n) - K(t^2 - t_n^2)/2\right]\right\} \int_{\frac{t+t_n-\tau_p-|t-t_n|}{2}}^{\frac{t+t_n+\tau_p-|t-t_n|}{2}} \exp\left[j2\pi K(t-t_n)t'\right] dt' \quad (11)$$

Let $t' - \frac{t+t_n}{2} = t''$, then

$$g(t) = \sum_n a_n \exp\left\{j2\pi\left[f_c(t-t_n) - K(t^2 - t_n^2)/2\right]\right\} \int_{\frac{\tau_p-|t-t_n|}{2}}^{\frac{\tau_p+|t-t_n|}{2}} \exp\left[j2\pi K(t-t_n)\left(t'' + \frac{t+t_n}{2}\right)\right] dt'' \\ = \sum_n a_n \exp\left[j2\pi f_c(t-t_n)\right] \int_{\frac{\tau_p-|t-t_n|}{2}}^{\frac{\tau_p+|t-t_n|}{2}} \exp\left[j2\pi K(t-t_n)t''\right] dt'' \quad (12)$$

The integration gives the sinc function:

$$g(t) = \sum_n a_n \exp\left[j2\pi f_c(t-t_n)\right] \cdot \frac{\sin\left(\tau_p - |t-t_n|\right) \pi K(t-t_n)}{\pi K(t-t_n)} \quad (13)$$

Remember it holds if $|t-t_n| < \tau_p$ otherwise $g(t) = 0$. The signal near to the maximum

power of the sinc function ($|t-t_n| \ll \tau_p$) can be rewritten as:

$$g(t) = \sum_n a_n \exp[j2\pi f_c(t-t_n)] \cdot \frac{\sin \tau_p \pi K(t-t_n)}{\pi K(t-t_n)} \quad (14)$$

Dropping the summation, the range compressed result of n^{th} pulse, $g_n(t)$ can be written as,

$$g_n(t) = a_n \exp[j2\pi f_c(t-t_n)] \cdot \frac{\sin \tau_p \pi K(t-t_n)}{\pi K(t-t_n)} \quad (15)$$

In other form using $t_n = nT_p + 2R_n/c$, $\omega = 2\pi f$ and $c = f\lambda$,

$$g_n(t) = a_n \exp(j\omega_c t) \exp(-j2\pi n T_p) \exp(-j4\pi R_n/\lambda) \cdot \frac{\sin \pi K \tau_p(t-t_n)}{\pi K(t-t_n)} \quad (16)$$

We assume synchronization of the detailed pulse waveform, that is the radar is coherent.

In effect, the term $\exp(-j2\pi n T_p)$ can be neglected as it is the same for all pulses. We

obtain:

$$g_n(t) = a_n \exp(j\omega_c t) \exp(-j4\pi R_n/\lambda) \cdot \frac{\sin \pi K \tau_p(t-t_n)}{\pi K(t-t_n)} \quad (17)$$

The envelope of this has a 3 dB width (half the maximum power)

$$\delta t = \frac{1}{K \tau_p} = \frac{1}{B_R} \quad (18)$$

This is the **slant range time resolution** of the SAR system, which corresponds to **slant range spatial resolution** (or, **range resolution** in short) as:

$$\delta R = c/2B_R, \quad (19)$$

where $B_R = K \tau_p$ is the range **chirp bandwidth**. Therefore, a large chirp bandwidth gives enhanced range resolution for a SAR system. Note that the range resolution holds with positive K , ‘up-chirp’. In case of ‘down-chirp’, K should be replaced with $|K|$.

3. Azimuth Compression

After range compression, the data is transformed into the azimuth frequency domain via FFT. There are two important parameters for azimuth compression: Doppler centroid frequency f_{Dc} and Doppler rate f_R . They can be derived from the satellite state vectors including satellite's position and velocity vectors. Sometimes, however, those values need to be recalculated more precisely, which can be done using the data itself. The Doppler centroid frequency f_{Dc} is determined from the data via the clutterlock procedure. The Doppler rate f_R can be determined by an autofocus procedure using the azimuth subaperture correlation method. Multiple pulse repetition frequency (PRF) ambiguity of Doppler centroid, which is important for range migration, can be solved by the range subaperture correlation method. Azimuth spectral filtering can be applied on two SAR signals forming an interferometric pair to compensate decorrelation from different Doppler centroids of the two images. The signal is then azimuth compressed using the same matched filtering as the case of range compression. A Hamming filter can then be applied to reduce the sidelobe effect similar to the one during the range compression. The following is the detailed procedures for azimuth compression.

Azimuth Matched Filtering

The angular carrier frequency ω_c from the range-compressed signal can be eliminated by demodulation. Selecting the value at $t = t_n$ that gives maximum $|g_n(t)|$ by precise sampling and representing t as slow time s gives:

$$\hat{g}(s|x_c, R_c) = \exp[-j4\pi R(s)/\lambda]. \quad (20)$$

The range function $R(s)$ can be expanded as a Taylor series around $s_c = x_c/V_s$, the slow time at which the centre of the radar beam crosses the target, where V_s is the speed of the radar platform relative to the target point.

$$R(s) = R_c + \dot{R}_c(s - s_c) + \ddot{R}_c(s - s_c)^2/2 + \dots \quad (21)$$

The Doppler frequency, by definition, is the time rate of phase $\phi(s)$

$$\phi(s) = -4\pi R(s)/\lambda, \quad (22)$$

$$f_D(s) = \dot{\phi}/2\pi = -2\dot{R}(s)/\lambda, \quad (23)$$

$$\dot{f}_D(s) = \ddot{\phi}/2\pi = -2\ddot{R}(s)/\lambda. \quad (24)$$

We can define the Doppler centroid and Doppler rate at $s = s_c$ as

$$f_{Dc} = -2\dot{R}_c/\lambda, \quad (25)$$

$$f_R = -2\ddot{R}_c/\lambda, \quad (26)$$

so that

$$R(s) = R_c - (\lambda f_{Dc}/2)(s - s_c) - (\lambda f_R/4)(s - s_c)^2. \quad (27)$$

The range-compressed signal can be expressed in terms of f_{Dc} and f_R as

$$\hat{g}(s|s_c, R_c) = \exp(-j4\pi R_c/\lambda) \exp\{j2\pi[f_{Dc}(s - s_c) + f_R(s - s_c)^2/2]\},$$

$$|s - s_c| < S/2, \quad (28)$$

where S is the azimuth integration time. This is a linear FM wave with centre frequency f_{Dc} and frequency rate f_R . The azimuth compression is to compute the correlation

$$\zeta(s) = \int_{-\infty}^{\infty} h^{-1}(s' - s) \hat{g}(s') ds'. \quad (29)$$

Similar to range compression, the azimuth compression can be realised approximately using a correlator function

$$h^{-1}(s) = \exp[-j2\pi(f_{Dc}s + f_R s^2/2)]. \quad (30)$$

The output of azimuth compression is then

$$\zeta(s | s_c, R_c) = \exp(j2\pi f_{Dc} s) \exp(-j4\pi R_c / \lambda) \cdot \frac{\sin \pi f_R S (s - s_c)}{\pi f_R (s - s_c)} \quad (31)$$

Derivation of this equation is similar to that of range compression by the following replacement of variables:

Range Compression	Azimuth Compression
t : fast time	s : slow time
t'	s'
t''	s''
t_n : two-way travel time	s_c : beam centre time
τ_p : pulse duration	S : coherent integration time
f_c : carrier frequency	f_{Dc} : Doppler centroid
K : chirp rate	f_R : Doppler rate

The peak of this pulse occurs at $s = s_c$, the target azimuth location. The 3 dB width of this pulse, that determines the **azimuth time resolution**, is

$$\delta s = 1/|f_R|S = 1/B_D \quad (32)$$

where $B_D = |f_R|S$ is the Doppler bandwidth. The **azimuth spatial resolution** (or, **azimuth resolution**) is then

$$\delta x = V_s \delta s = V_s / B_D = V_s / |f_R|S. \quad (33)$$

From the simple geometry of a radar antenna, with physical length L_a along track, the nominal beam width is $\theta_H = \lambda/L_a$ so that any particular target point at range R_c is illuminated for a nominal time $S = \lambda R_c / V_s L_a$.

For the geometry of the figure below, given a squint angle θ_s , the Doppler parameters can be geometrically determined as

$$R^2(s) = R_c^2 + V_s^2(s - s_c)^2 - 2R_c V_s (s - s_c) \sin \theta_s \quad (34)$$

$$R(s) \approx R_c + V_s^2(s - s_c)^2 / 2R_c - V_s(s - s_c) \sin \theta_s \quad (35)$$

$$f_{Dc} = (2V_s / \lambda) \sin \theta_s, \quad (36)$$

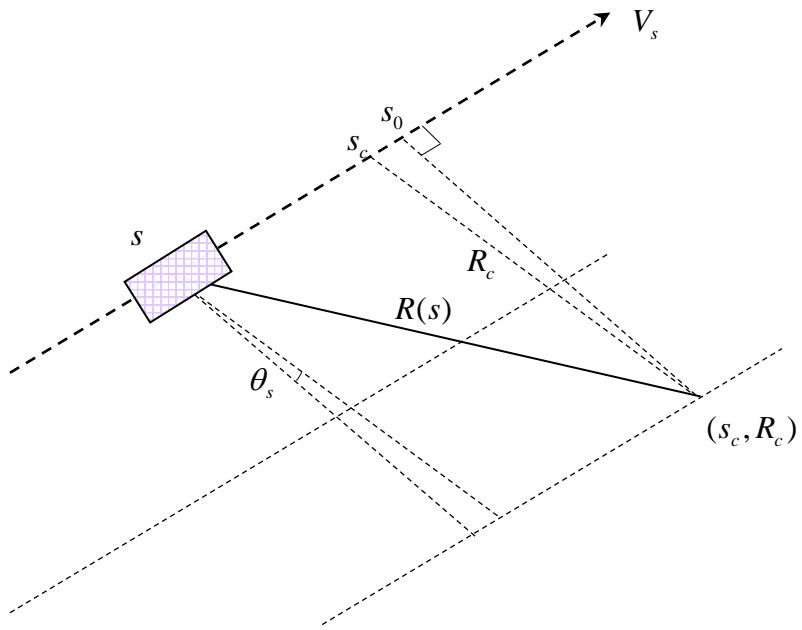
$$f_R = -2V_s^2 / \lambda R_c. \quad (37)$$

In case of a real aperture radar, azimuth resolution changes inversely with the physical length of its antenna as $\delta x = \lambda R_c / L_a$. For SAR, the Doppler bandwidth is therefore $B_D = 2V_s / L_a$ and the system azimuth resolution is

$$\delta x = L_a / 2. \quad (38)$$

δx changes in proportion to the physical length of SAR antenna. (39)

With intensive signal compression processing, high azimuth resolution can be achieved with a SAR in a significantly smaller physical dimension than that in a real aperture radar.



Appendix A.

Dirac Delta Function

$$\delta(x) = 0, x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} f(x) \delta(x) = f(0)$$

No such function exists.

Sinc function – one approximate of Dirac Delta Function

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^n e^{ixt} dt$$

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1$$

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0)$$

$$\text{sinc}(x) = \frac{\sin x}{x}$$

Appendix B.

SAR Focusing – Point Target

azimuth

↓
↓
↓
↓

